CEN Tech. J., vol. 7(2), 1993, pp.151–154

Plimpton 322: A Sample of Ancient Mathematics

DAVID MALCOLM

There is an ancient Babylonian clay tablet, identified as Plimpton 322, at Columbia University. The code name identifies it as catalogue item 322 in the G. A. Plimpton collection. Figure 1 shows a representation of this clay tablet, which is written in Old Babylonian script, so it is conventionally dated at between 1900 and 1600 BC.

The tablet shows a high level of mathematical ability at a very early date, as we will try to explain.

Perhaps the reader would like to try to understand what this tablet contains without being told, so we will start off by explaining how to read the numbers on it without actually giving the interpretation placed on it by modern mathematicians.

CUNEIFORM SYMBOLS

This particular cuneiform language is well understood; and the tablet contains mainly numbers arranged in columns. The boomerang symbol represents 10, and the vertical trumpet represents 1. Stylised combinations of boomerangs represent 20 through 50, while combinations of trumpets represent 2 through 9. For example, the first entry in column I is a 1 followed by 59 followed by 15. It is also a well-known fact that the Babylonians used a sexigesimal number system (that is, base 60) so that the first number in column II (a 1 followed by a 59) represents 1 x 60 + 59, or 119; and the first number in column III (a 2 followed by 49) represents 2 x 60 + 49, or 169. The numbers in column IV are simply line numbers running from 1 to 15.

The tablet Plimpton 322 is depicted in the literature,[1] which is slightly different from what we show in Figure 1:—

(1) Areas that are damaged on the actual tablet are here reconstructed.

(2) Four mistakes made by the Babylonian mathematician,

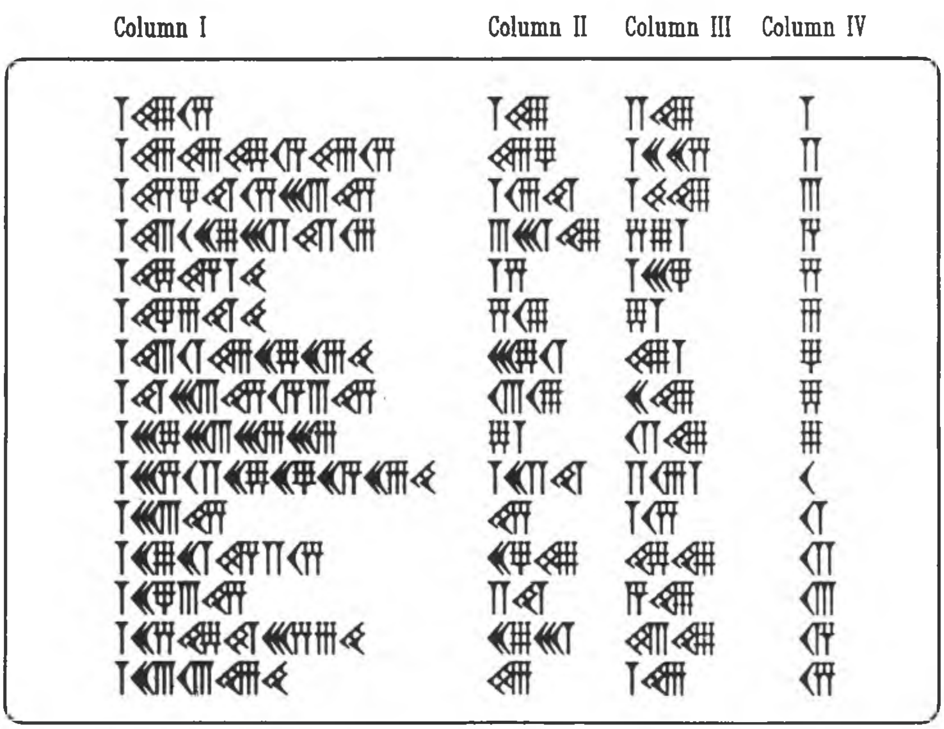

Figure 1. *A representation of clay tablet Plimpton 322, written in Old Babylonian cuneiform script.*

as reported in the literature,[2] are shown here corrected.

(3) A fifth mistake, not reported in the literature, which occurs in column I is also shown corrected.

(4) The four columns of numbers have actual headings, which the experts have not been able to translate exactly, but both column I and column III contain a word which translates to 'diagonal'.

It should also be noted that the Babylonians had no zero. In fact the number in line 1 column I is 1,59,0,15 and the number in line 13 column I is 1,27,0,3,45 (just using a comma to separate between the numbers). Neither did they have an equivalent of a decimal point; the first column should be read as 1 followed by a series of decreasing fractions.

Now we should give due credit to the mathematicians who deciphered this clay tablet.[3] Who would have guessed that the numbers are mathematical? They could have just been part of a shopping list, or perhaps a price list.

CORRECTIONS

As noted above, five mistakes on the original tablet have been corrected in Figure 1. We will explain these as a final clue as to how the ancient mathematicians may have arrived at Plimpton 322.

(1) In line 9 column II, the tablet contains 9,1 whereas it should contain 8,1. It is presumed this is a simple transcription error.

(2) In line 13 column II, the tablet contains 7,12,1 which is the square of the correct figure of 2,41.

(3) In line 15 column III, the tablet contains 53 which is half of the correct figure of 1,46.

(4) In line 2 column III, the tablet contains 3,12,1 instead of the correct figure of 1,20,25. One fairly complex explanation has been offered as to how this mistake may have been made.[4]

(5) In line 8 column I the tablet contains 1,41,33,59,3,45 whereas the correct figure is 1,41,33,45,14,3,45. It looks like a carry has occurred into the wrong column.[5]

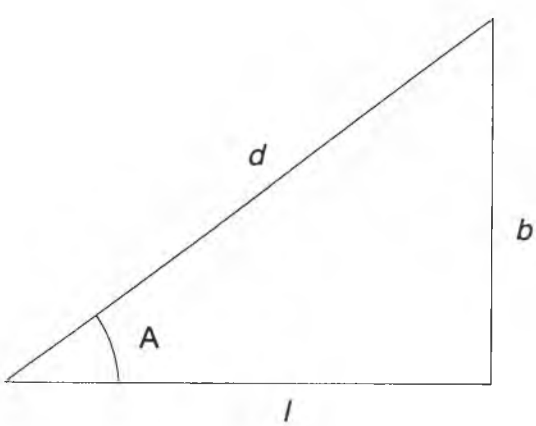

Figure 2. A right-angled triangle.

PYTHAGORAS

So what do the numbers mean? First let's recall Pythagoras' theorem. The square on the hypotenuse of a right-angled triangle equals the sum of the squares on the other two sides. In terms of the triangle shown in Figure 2, that is:—

$$d^2 = l^2 + b^2$$

Now suppose that we wanted to obtain a number of such triangles in which l, b and d all have integral values. Very many such triangles exist, but most of us probably only know of one — the well-known 3,4,5 triangle. Which shows that there was some scholar in ancient Babylon who was more knowledgeable than us.

Plimpton 322 is in fact a list of 15 different triangles of this type, with angle A in the range of 30 to 45 degrees. Each line refers to a different such triangle. Column II is b and column III is d, while l is not shown. The tablet does show evidence of having been broken along the left-hand edge, so perhaps the column of l values has been lost. Line 11 is in fact the familiar 3,4,5 triangle, except that each side has been multiplied by 15. (It is the only one which has had each side multiplied by a common factor.)

TRIGONOMETRY

So each line represents a different right-angled triangle with integral sides, and angle A in the range of 30 to 45 degrees. And they are arranged in order of decreasing angle A, or of (d/l) squared, which is what column I contains. Howard Eves, in his book,[6] observes that this is the square of the secant of angle A, thus introducing the thought that this relates to trigonometry.

The fact that the word 'diagonal' occurs in the column headings makes it clear that whoever put it together knew that it was a geometric problem as well as a problem in number theory. We quote the authority Otto Neugebauer as to the translation of these headings:—

> *'Columns II and III are headed by words which might be translated as "solving number of the width" and "solving number of the diagonal" respectively. "Solving number" is a rather unsatisfactory rendering for a term which is used in connection with square roots and similar operations and has no exact equivalent in our modern terminology . . . The word "diagonal" occurs also in the heading of the first column but the exact meaning of the remaining words escapes us.'*[7]

Unlike our modern decimal system, the Babylonians used exact relations. In the fifth line for example, the value of the secant squared of A is given as:—

$$1 + 48/60 + 54/(60)^2 + 1/(60)^3 + 40/(60)^4$$

which is the exact answer (for you to check, the sides are 72, 65 and 97). In the case of these triangles, how did they manage to get exact answers for all the values of $\sec^2(A)$? Not only did the Babylonians know Pythagoras' theorem, well before the time of Pythagoras, but they were also apparently familiar with the parametric form of Pythagorean triples[8,9]:—

$$l = 2pq$$
$$b = p^2 - q^2$$
$$d = p^2 + q^2$$

where p and q are arbitrary integers, subject only to the conditions that they are relatively prime, not both odd, and p is greater than q. Additionally, for the Babylonian system they could only contain the prime factors 2, 3 and 5. On this basis it is clear that l, the denominator of the required function, can only have prime factors of 2, 3 and 5, so an exact finite result will always be obtained with sexagesimal arithmetic.

Now the question arises as to why these particular triangles have been chosen, and what other triangles exist? In fact there are surprisingly few others. With the benefit of a digital computer, a program was written to search for triangles with angle A falling in the range of 30 to 45 degrees. The result was that there are only three other such triangles.[10] Figure 3 gives the complete table. We limited the maximum values of p and q to 125, because the standard tables of reciprocals used by the Babylonian mathematicians only went up to 81, although 125 was 'well known as the canonical example for the computation of reciprocals beyond the standard table'.[11] Of the three extra triangles, two have $p = 125$ which may have been difficult for them to find, and the other one is strictly outside of the range of A as recorded on the tablet (p=16, q=9 gives angle A as 31:17:04, which is lower than 31:53:27, the last entry on the tablet). So they didn't miss out many at all.

CONCLUSIONS

This particular tablet has been recognised for its mathematical value, but one can't help wondering if other records exist of equal or greater significance in museum collections somewhere.

So in summing up, it is clear that the Babylonians had an advanced knowledge of mathematics, and Pythagoras' theorem was known a thousand years before Pythagoras.[12] The mathematics we have looked at is not easy for us to comprehend, which should in itself cause us to be impressed with the mathematical knowledge of the ancient Babylonians.

Authors with an evolutionary outlook use words like 'most remarkable' and 'truly remarkable' to describe the knowledge evidenced by this clay tablet.[13] But it should hardly be surprising to those who take the Bible seriously. We don't see man as increasing in intelligence, and we are led to believe that Babylon was the centre of world learning after the time of the Flood. Our authority Eves goes on in his next section to discuss the mathematics of Egypt, and says:—

'The mathematics of Egypt, contrary to much popular

ANGLE A	p	q	l	b	d	COLUMN I	COLUMN II	COLUMN III
44:45:37	12	5	120	119	169	1 59 0 15	1 59	2 49
44:15:10	64	27	3456	3367	4825	1 56 56 58 14 50 6 15	56 7	1 20 25
43:47:14	75	32	4800	4601	6649	1 55 7 41 15 33 45	1 16 41	1 50 49
43:16:17	125	54	13500	12709	18541	1 53 10 29 32 52 16	3 31 49	5 9 1
42:04:30	9	4	72	65	97	1 48 54 1 40	1 5	1 37
41:32:40	20	9	360	319	481	1 47 6 41 40	5 19	8 1
40:18:55	54	25	2700	2291	3541	1 43 11 56 28 26 40	38 11	59 1
39:46:13	32	15	960	799	1249	1 41 33 45 14 3 45	13 19	20 49
38:43:05	25	12	600	481	769	1 38 33 36 36	8 1	12 49
37:26:14	81	40	6480	4961	8161	1 35 10 2 28 27 24 26 40	1 22 41	2 16 1
36:52:12	2	1	4	3	5	1 33 45	3	5
35:46:30	125	64	16000	11529	19721	1 31 9 9 25 42 2 15	3 12 9	5 28 41
34:58:34	48	25	2400	1679	2929	1 29 21 54 2 15	27 59	48 49
33:51:18	15	8	240	161	289	1 27 0 3 45	2 41	4 49
33:15:43	50	27	2700	1771	3229	1 25 48 51 35 6 40	29 31	53 49
31:53:27	9	5	90	56	106	1 23 13 46 40	56	1 46
31:17:04	16	9	288	175	337	1 22 9 12 36 15	2 55	5 37
30:06:58	125	72	18000	10441	20809	1 20 11 16 19 14 24	2 54 1	5 46 49

Figure 3. *The complete table of Pythagorean triples for triangles with angle A falling in the range 30–45° (left), compared with the corrected Babylonian numbers as translated from Plimpton 322 (right), which can then be compared with the cuneiform on the representation in Figure 1.*

opinion, never reached the level attained by Babylonian mathematics'.[14]

Let us just consider one last thought. Why does the tablet contain so many errors? Of the 15 triangles, at least five contained an error. Perhaps Plimpton 322 was just an assignment done by a student, and in fact the knowledge of the experts far exceeded the level of knowledge shown on this clay tablet.

REFERENCES

1. Eves, H. W., 1976. **An Introduction to the History of Mathematics**, 4th edition, Holt Rinehart and Winston, p. 36.
2. Neugebauer, O., 1957. **The Exact Sciences of Antiquity**, 2nd edition, Brown University Press, p. 37.
3. Eves, Ref. 1, p. 35 — it was first reported by Neugebauer and Sachs in 1945.
4. Gillings, R. J., 1953. Unexplained error in Babylonian cuneiform tablet. **The Australian Journal of Science**, October 1953, pp. 54–56.
5. By generating the table by computer, it was easy to check all figures.
6. Eves, Ref. 1, p. 38.
7. Neugebauer, Ref. 2, p. 37.
8. Neugebauer, Ref. 2, p. 39.
9. Eves, Ref. 1, p. 37.
10. If p is limited to a maximum of 125, and q is less than p, then we can obtain a total of 152 Pythagorean triples. Of these, 18 have an angle A in the range 30 to 45 degrees. Fortran 77 program available on request.
11. Neugebauer, Ref. 2, p. 39.
12. Neugebauer, Ref. 2, p. 36.
13. Eves, Ref. 1, 'most remarkable' — p. 35, 'truly remarkable' — p. 38.
14. Eves, Ref. 1, p. 38.

David Malcolm has a Masters Degree in Engineering and works as a Computer Systems Officer for the School of Engineering at The University of Newcastle, Newcastle, Australia.